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Quantum effective potential for $U(1)$ fields on $\mathrm{S}^2_L \times \mathrm{S}^2_L$

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ABSTRACT: We compute the one-loop effective potential for noncommutative $U(1)$ gauge fields on $\mathbf{S}^2_L \times \mathbf{S}^2_L$. We show the existence of a novel phase transition in the model from the 4−dimensional space $\mathbf{S}^2_L \times \mathbf{S}^2_L$ to a matrix phase where the spheres collapse under the effect of quantum fluctuations. It is also shown that the transition to the matrix phase occurs at infinite value of the gauge coupling constant when the mass of the two normal components of the gauge field on $\mathbf{S}^2_L \times \mathbf{S}^2_L$ is sent to infinity.

KEYWORDS: Fuzzy sphere, NC gauge theory, matrix model, effective action, phase structure.

Contents

1. Introduction

Fuzzy approximations of spacetime (like lattice regularizations) are designed for the study of gauge theories in the nonperturbative regime using Monte-Carlo simulations. They consist in replacing continuous manifolds by matrix algebras. The resulting field theory will thus only have a finite number of degrees of freedom and hence it is regularized. The claim is that this method has the advantage -in contrast with lattice- of preserving all continuous symmetries of the original action at the classical level [5, 6, 7].

Field theory on the fuzzy sphere is the most studied example in the literature. In perturbation theory it is shown that scalar field theories on $\mathbf{S}^2_L$ suffer from the UV-IR mixing problem [8]. Moreover it is shown that there exists new nonperturbative phenomena which are missing in the commutative theory. For example a novel phase has been discovered in scalar field theories on $\mathbf{S}^2_L$ (the so-called non-uniform phase or matrix phase) which has no commutative analogue [9]. This new phase was also observed in three dimensions [10]. Generalization to 4$-$dimensional fuzzy spaces and their scalar field theories were undertaken in [11].

The quantum properties of the gauge field on the fuzzy sphere have been studied in [1, 2, 3]. In [1] the effective action was computed to one loop for $U(1)$ gauge fields. It was shown that the model contains a gauge invariant UV-IR mixing in the limit $L \to \infty$, i.e the effective action does not go over to the commutative action in the continuum limit. Furthermore a first order phase transition was observed at one-loop from the fuzzy sphere phase to a matrix phase where the sphere collapses. This transition was previously detected in Monte Carlo simulation of the model reported in [2]. In some sense the one-loop result for the $U(1)$ model is exact.

It was also shown in [1] that the UV-IR mixing and the matrix phase both disappear in the limit where we send the mass of the normal scalar component of the gauge field on $\mathbf{S}^2_L$ to infinity. This means in particular that the nonperturbative $\mathbf{S}^2_L$-to-matrix phase transition is a reflection of the UV-IR mixing seen in perturbation theory and that this latter finds its origin in the coupling of the normal scalar field to the two dimensional gauge field. The differential calculus on the fuzzy sphere is intrinsically $3-$dimensional and as a consequence there is no a gauge-covariant splitting of the $3-$dimensional fuzzy gauge field into its normal and tangent components on $\mathbf{S}^2_L$; hence the action will necessarily involve the interaction of the two fields. This result (among many others) was confirmed recently in our Monte Carlo simulation of the model where we have also found a novel third-order one-plaquette-like phase transition which the model undergoes and which we can also trace to the coupling of the normal scalar field. The full phase diagram of the model will be reported elsewhere [4].

The main goal of this article is to study the phase structure of $U(1)$ gauge theories on fuzzy $\mathbf{S}^2_L \times \mathbf{S}^2_L$. The advantage of considering $\mathbf{S}^2_L \times \mathbf{S}^2_L$ is that one can use all the machinery of the well known $SU(2)$ polarization tensors. Other studies of noncommutative gauge theories on $4-$dimensional fuzzy spaces have already appeared [12].

This article is organized as follows. In section 2 we give a brief description of the geometry of fuzzy $\mathbf{S}^2_L \times \mathbf{S}^2_L$. In section 3 we introduce fuzzy gauge fields and we write down the action we will study in this article. In section 4 we compute the effective potential. In section 5 we show the existence of a first order $\mathbf{S}^2_L \times \mathbf{S}^2_L$-to-matrix phase transition in exact analogy with the two-dimensional case and we derive the critical line. Section 6 contains the conclusion.

2. Fuzzy $\mathbf{S}^2_L \times \mathbf{S}^2_L$

Fuzzy $\mathbf{S}^2_L \times \mathbf{S}^2_L$ is the simpliest 4 dimensional fuzzy space. It is a finite dimensional matrix approximation of the cartesian product of two continuous spheres. This fuzzy space is defined by a sequence of Connes triples [13]

$$\mathbf{S}^2_L \times \mathbf{S}^2_L = \left\{ Mat_{(L+1)^2}, H_L, \Delta_L \right\}. \tag{2.1}$$

$Mat_{(L+1)^2}$ is the matrix algebra of dimension $(L+1)^2$ and Δ_L is a suitable Laplacian acting on matrices which encodes the geometry of the space. It is defined by

$$\Delta_L \equiv [L_{AB}, [L_{AB}, \cdot\ ]] = \mathcal{L}^2_{AB} \tag{2.2}$$

where L_{AB}, with $A, B = \overline{1,4}$, are the generators of the irreducible representation $(\frac{L}{2}, \frac{L}{2})$ of $SO(4)$. The generators L_{AB} (with $L_{AB} = -L_{BA}$) satisfy the commutation relations

$$\begin{aligned} [L_{AB}, L_{CD}] &= f_{ABCDEF} L_{EF} \\ &\equiv \delta_{BC} L_{AD} - \delta_{BD} L_{AC} + \delta_{AD} L_{BC} - \delta_{AC} L_{BD}. \end{aligned} \tag{2.3}$$

H_L in (2.1) is the Hilbert space (with inner product $< M, N >= \frac{1}{(L+1)^2} Tr\left(M^\dagger N\right)$) which is associated with the irreducible representation $(\frac{L}{2}, \frac{L}{2})$ of $SO(4)$.

Since $SO(4) = [SU(2)\times SU(2)]/Z_2$ we can introduce $SU(2)$ (mutually commuting) generators $L_a^{(1)}$ and $L_a^{(2)}$ by $-2L_a^{(1)} = \frac{1}{2}\epsilon_{abc}L_{bc} + L_{a4}$ and $-2L_a^{(2)} = \frac{1}{2}\epsilon_{abc}L_{bc} - L_{a4}$ with $a = 1,2,3$ and ϵ_{abc} is the three dimensional Levi-Civita tensor. Then it can be easily shown that the two $SO(4)$ quadratic Casimir can be rewritten in the form (where ϵ_{ABCD} is the four dimensional Levi-Civita tensor)

$$\begin{aligned} L_{AB}^2 &= 4[(L_a^{(1)})^2 + (L_a^{(2)})^2] = 2L(L+2)\equiv 8c_2 \\ \epsilon_{ABCD}L_{AB}L_{CD} &= 8[(L_a^{(1)})^2 - (L_a^{(2)})^2]\equiv 0. \end{aligned} \tag{2.4}$$

Similarly the Laplacian $\mathcal{L}_{AB}^2$ reads in terms of the three dimensional indices as follows

$$\mathcal{L}_{AB}^2 = 4\left[\left(\mathcal{L}_a^{(1)}\right)^2 + \left(\mathcal{L}_a^{(2)}\right)^2\right], \tag{2.5}$$

where $\mathcal{L}_a^{(1)} \equiv [L_a^{(1)}, \cdot\,]$ and $\mathcal{L}_a^{(2)} \equiv [L_a^{(2)}, \cdot\,]$. For $\mathbf{S}_L^2 \times \mathbf{S}_L^2$ the algebra $Mat_{(L+1)^2}$ is generated by the coordinate operators

$$x_a^{(1)} = R_1\frac{L_a^{(1)}}{\sqrt{c_2}}, \qquad x_a^{(2)} = R_2\frac{L_a^{(2)}}{\sqrt{c_2}} \tag{2.6}$$

which satisfy

$$\sum_{a=1}^{3}\left(x_a^{(i)}\right)^2 = R_i^2\mathbf{1}, \qquad \left[x_a^{(i)}, x_b^{(j)}\right] = \frac{i\,R_i}{\sqrt{c_2}}\delta_{ij}\epsilon_{abc}x_c^{(i)}, \qquad i = 1,2. \tag{2.7}$$

In the limit $L \to \infty$ keeping R_1 and R_2 fixed we recover the commutative algebra of functions on $\mathbf{S}^2 \times \mathbf{S}^2$. If we also choose to scale the radii R_1 and R_2 such as for example $\theta_1^2 = R_1^2/L_1$ and $\theta_2^2 = R_2^2/L_2$ are kept fixed we obtain the non-commutative Moyal-Weyl space $\mathbb{R}^2_{\theta_1} \times \mathbb{R}^2_{\theta_2}$ [11].

The algebra of matrices $Mat_{(L+1)^2}$ can be decomposed under the action of the two $SU(2)$ of $SO(4)$ as $Mat_{(L+1)} \otimes Mat_{(L+1)}$. As a consequence a general function on $\mathbf{S}_L^2 \times \mathbf{S}_L^2$ can be expanded in terms of polarization tensors [14] as follows

$$\phi = \sum_{k_1=0}^{L}\sum_{m_1=-k_1}^{k_1}\sum_{k_2=0}^{L}\sum_{m_2=-k_2}^{k_2}\phi_{k_1m_1k_2m_2}\hat{Y}_{k_1m_1}\otimes\hat{Y}_{k_2m_2}. \tag{2.8}$$

3. Fuzzy gauge fields

$U(n)$ gauge field on $\mathbf{S}_L^2\times\mathbf{S}_L^2$ can be associated with a set of six hermitian matrices $D_{AB} \in Mat_{n(L+1)^2}$ ($D_{AB} = -D_{BA}$) which transform homogeneously under the action of the group, i.e

$$D_{AB} \to UD_{AB}U^{-1}, \qquad U \in U\left(n(L+1)^2\right). \tag{3.1}$$

In this paper we will be mainly interested in $U(1)$ theory on $S_L^2 \times S_L^2$. The action is given by (with $Tr_L = \frac{1}{(L+1)^2}Tr$, g is the gauge coupling constant and m is the mass of the normal components of the gauge field)

$$S = \frac{1}{16g^2}\left\{-\frac{1}{4}Tr_L[D_{AB}, D_{CD}]^2 + \frac{i}{3}f_{ABCDEF}Tr_L[D_{AB}, D_{CD}]D_{EF}\right\}$$

$$+ \frac{m^2}{8g^2L_{AB}^2}Tr_L(D_{AB}^2 - L_{AB}^2)^2 + \frac{m^2}{32g^2L_{AB}^2}Tr_L(\epsilon_{ABCD}D_{AB}D_{CD})^2. \qquad (3.2)$$

The equations of motion are given by

$$i[D_{CD}, F_{AB,CD}] + \frac{4m^2}{\sqrt{c_2}}\{D_{AB}, \Phi_1 + \Phi_2\} + \frac{m^2}{\sqrt{c_2}}\{\epsilon_{ABCD}D_{CD}, \Phi_1 - \Phi_2\} = 0. \qquad (3.3)$$

As we will see shortly $F_{AB,CD} = i\,[D_{AB}, D_{CD}] + f_{ABCDEF}D_{EF}$ can be interpreted as the curvature of the gauge field on fuzzy $\mathbf{S}_L^2 \times \mathbf{S}_L^2$ whereas Φ_1 and Φ_2 (defined by $D_{AB}^2 - L_{AB}^2 = 8\sqrt{c_2}(\Phi_1 + \Phi_2)$ and $\epsilon_{ABCD}D_{AB}D_{CD} = 16\sqrt{c_2}(\Phi_1 - \Phi_2)$) can be interpreted as the normal components of the gauge field on $\mathbf{S}_L^2 \times \mathbf{S}_L^2$.

The most obvious non-trivial solution of the equations of motion (3.3) must satisfy $F_{AB,CD} = 0$, $D_{AB}^2 = L_{AB}^2$ and $\epsilon_{ABCD}D_{AB}D_{CD} = 0$ (or equivalently $F_{AB} = 0$, $\Phi_i = 0$). This solution is clearly given by the generators L_{AB} of the irreducible representation $(\frac{L}{2}, \frac{L}{2})$ of $SO(4)$, viz

$$D_{AB} = L_{AB}. \qquad (3.4)$$

As it turns out this is also the absolute minimum of the model. By expanding D_{AB} around this vacuum as $D_{AB} = L_{AB} + A_{AB}$ and substituting back into the action (3.2) we obtain a $U(1)$ gauge field A_{AB} on $\mathbf{S}_L^2 \times \mathbf{S}_L^2$ with the correct transformation law under the action of the group, namely $A_{AB} \longrightarrow UA_{AB}U^{-1} + U\mathcal{L}_{AB}U^{-1}$. The matrices D_{AB} are thus the covariant derivatives on $\mathbf{S}_L^2 \times \mathbf{S}_L^2$. The curvature $F_{AB,CD}$ in terms of A_{AB} takes the usual form $F_{AB,CD} = i\mathcal{L}_{AB}A_{CD} - i\mathcal{L}_{CD}A_{AB} + f_{ABCDEF}A_{EF} + i[A_{AB}, A_{CD}]$. The normal scalar fields in terms of A_{AB} are on the other hand given by $8\sqrt{c_2}(\Phi_1 + \Phi_2) = L_{AB}A_{AB} + A_{AB}L_{AB} + A_{AB}^2$ and $16\sqrt{c_2}(\Phi_1 - \Phi_2) = \epsilon_{ABCD}(L_{AB}A_{CD} + A_{AB}L_{CD} + A_{AB}A_{CD})$.

We can verify this conclusion explicitly by introducing the matrices $D_a^{(1)} = L_a^{(1)} + A_a^{(1)}$ and $D_a^{(2)} = L_a^{(2)} + A_a^{(2)}$ defined by

$$D_a^{(1)} \equiv -\frac{1}{2}\left[\frac{1}{2}\epsilon_{abc}D_{bc} + D_{a4}\right]\ ,\ D_a^{(2)} \equiv -\frac{1}{2}\left[\frac{1}{2}\epsilon_{abc}D_{bc} - D_{a4}\right]. \qquad (3.5)$$

Clearly $D_a^{(1)}$ ($A_a^{(1)}$) and $D_a^{(2)}$ ($A_a^{(2)}$) are the components of D_{AB} (A_{AB}) on the two spheres respectively. The curvature becomes $F_{ab}^{(i,j)} = i\mathcal{L}_a^{(i)}A_b^{(j)} - i\mathcal{L}_b^{(j)}A_a^{(i)} + \delta_{ij}\epsilon_{abc}A_c^{(i)} + i[A_a^{(i)}, A_b^{(j)}]$ whereas the normal scalar fields become $2\sqrt{c_2}\Phi_i = (D_a^{(i)})^2 - c_2 = L_a^{(i)}A_a^{(i)} + A_a^{(i)}L_a^{(i)} + (A_a^{(i)})^2$. In terms of this three dimensional notation the action (3.2) reads

$$S = S^{(1)} + S^{(2)} + \frac{1}{2g^2}Tr_L\left(F_{ab}^{(1,2)}\right)^2. \qquad (3.6)$$

$S^{(1)}$ and $S^{(2)}$ are the actions for the $U(1)$ gauge fields $A_a^{(1)}$ and $A_a^{(2)}$ on a single fuzzy sphere $\mathbf{S}_L^2$. They are given by

$$S^{(i)} = \frac{1}{4g^2}Tr_L\left(F_{ab}^{(i,i)}\right)^2 - \frac{1}{2g^2}\epsilon_{abc}Tr_L\left[\frac{1}{2}F_{ab}^{(i,i)}A_c^{(i)} - \frac{i}{6}[A_a^{(i)}, A_b^{(i)}]A_c^{(i)}\right] + \frac{2m^2}{g^2}Tr_L\Phi_i^2. \qquad (3.7)$$

It is immediately clear that in the continuum limit $L\longrightarrow\infty$ the action (3.6) describes the interaction of a genuine 4−d gauge field with the normal scalar fields $\Phi_i = n_a^{(i)}A_a^{(i)}$ where $n_a^{(i)}$ is the unit normal vector to the i-th sphere. The parameter m is precisely the mass of these scalar fields. Let us also remark that in this limit the 3−dimensional fields $A_a^{(i)}$ decompose as $A_a^{(i)} = (A_a^{(i)})^T + n_a^{(i)}\Phi_i$ where $(A_a^{(i)})^T$ are the tangent 2−dimensional gauge fields. Since the differential calculus on $\mathbf{S}^2_L \times \mathbf{S}^2_L$ is intrinsically 6−dimensional we can not decompose the fuzzy gauge field in a similar (gauge-covariant) fashion and as a consequence we can not write an action on the fuzzy $\mathbf{S}^2_L \times \mathbf{S}^2_L$ which will only involve the desired 4−dimensional gauge field.

4. Quantum effective potential

The partition function of the theory depends on 3 parameters, the Yang-Mills coupling constant g, the mass m of the normal scalar fields, and the size L of the matrices, viz

$$Z_L\left[J,g,m\right] = \int \prod_{A<B=1}^{4} \left[dX_{AB}\right] e^{-S[X]+Tr_L[J_{AB}X_{AB}]}. \tag{4.1}$$

In the background field method the field is decomposed as $X_{AB} = D_{AB} + Q_{AB}$ where D_{AB} is the background we are interested in studying and Q_{AB} stands for the fluctuation field. We add the usual gauge fixing and Faddeev-Popov terms given by

$$S_{g.f} + S_{gh} = -\frac{1}{32g^2}Tr_L\frac{[D_{AB},Q_{AB}]^2}{\xi} + \frac{1}{16g^2}Tr_L c[D_{AB},[D_{AB},b]]. \tag{4.2}$$

Performing the Gaussian path integral we obtain the one-loop effective action

$$\Gamma\left[D_{AB}\right] = S\left[D_{AB}\right] + \frac{1}{2}Tr_6 TR\log\Omega_{ABCD} - TR\log\mathcal{D}^2_{AB}. \tag{4.3}$$

Ω_{ABCD} is defined by

$$\Omega_{ABCD} = \frac{1}{2}\mathcal{D}^2_{EF}\delta_{AB,CD} - \left(1-\frac{1}{\xi}\right)\mathcal{D}_{AB}\mathcal{D}_{CD} - 2i\mathcal{F}_{ABCD} + \frac{4m^2}{L^2_{AB}}\Omega^{(1)}_{ABCD}, \tag{4.4}$$

where $\delta_{AB,CD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}$, and

$$\begin{aligned}\Omega^{(1)}_{ABCD} &= (D^2_{EF} - L^2_{EF})\delta_{AB,CD} + \frac{1}{2}(\epsilon_{EFGH}D_{EF}D_{GH})\epsilon_{ABCD}\\ &\quad -\mathcal{D}_{AB}\mathcal{D}_{CD} - \widetilde{\mathcal{D}}_{AB}\widetilde{\mathcal{D}}_{CD} + 4D_{AB}D_{CD} + 4\widetilde{D}_{AB}\widetilde{D}_{CD}.\end{aligned} \tag{4.5}$$

The notation $\mathcal{D}_{AB}$ and $\mathcal{F}_{ABCD}$ means that the covariant derivative D_{AB} and the curvature F_{ABCD} act by commutators, i.e $\mathcal{D}_{AB}(M) = [D_{AB},M]$, $\mathcal{F}_{ABCD}(M) = [F_{ABCD},M]$ where M is an element of $Mat_{(L+1)^2}$. Wwe have also introduced the notation $\widetilde{D}_{AB} \equiv \frac{1}{2}\epsilon_{ABCD}D_{CD}$. TR is the trace over the 4 indices corresponding to the left and right actions of operators on matrices. Tr_6 is the trace associated with the action of $SU(2)\times SU(2)$.

The main goal of this article is to check the stability of the solution (3.4), in other words to check whether or not the fuzzy space $\mathbf{S}^2_L\times\mathbf{S}^2_L$ is stable under quantum fluctuations.

Towards this end it is sufficient to consider only the background field $D_{AB}=\phi L_{AB}$ where the order parameter ϕ plays the role of the radius of the two spheres of $\mathbf{S}^2_L\times\mathbf{S}^2_L$. Therefore the computation of the effective action reduces to the computation of the effective potential $V_{\rm eff}(\phi)\equiv\Gamma[\phi L_{AB}]$. The classical potential is given by

$$V{\equiv}S[\phi L_{AB}]=\frac{L(L+2)}{g^2}\left(\frac{1}{4}\phi^4-\frac{1}{3}\phi^3+\frac{1}{4}m^2\left(\phi^2-1\right)^2\right). \tag{4.6}$$

The effective potential (in the gauge $\xi=1$) is given by

$$\begin{aligned}V_{\rm eff}&=V+\frac{1}{2}Tr_6TR\log\phi^2-TR\log\phi^2+\frac{1}{2}Tr_6TR\log\tilde{\Omega}_{ABCD}\\&=V+4(L+1)^4\log\phi+\frac{1}{2}Tr_6TR\log\tilde{\Omega}_{ABCD}.\end{aligned} \tag{4.7}$$

We are only interested in the $\phi-$dependence of the operator $\tilde{\Omega}$ which is defined by

$$\tilde{\Omega}_{ABCD}=\frac{1}{2}\mathcal{L}^2_{EF}\delta_{AB,CD}+2i\left(1-\frac{1}{\phi}\right)f_{ABCDEF}\mathcal{L}_{EF}+\frac{4m^2}{L^2_{AB}}\tilde{\Omega}^{(1)}_{ABCD}, \tag{4.8}$$

where

$$\tilde{\Omega}^{(1)}_{ABCD}=\left(1-\frac{1}{\phi^2}\right)L^2_{EF}\delta_{AB,CD}-\mathcal{L}_{AB}\mathcal{L}_{CD}-\widetilde{\mathcal{L}}_{AB}\widetilde{\mathcal{L}}_{CD}+4L_{AB}L_{CD}+4\widetilde{L}_{AB}\widetilde{L}_{CD}. \tag{4.9}$$

We will need to use the following identities

$$\begin{aligned}X_{AB}Y_{AB}&=4\left(X^{(1)}_aY^{(1)}_a+X^{(2)}_aY^{(2)}_a\right),\\f_{ABCDEF}Tr\left[X_{AB}Y_{CD}Z_{EF}\right]&=16\epsilon_{abc}Tr\left[X^{(1)}_aY^{(1)}_bZ^{(1)}_c+X^{(2)}_aY^{(2)}_bZ^{(2)}_c\right].\end{aligned} \tag{4.10}$$

The matrices $X^{(i)}_a$ ($Y^{(i)}_a$) are related to the matrices X_{AB} (Y_{AB}) by equations of the form (3.5). Using these identities we can express the last term in (4.7) in the following way

$$\begin{aligned}\frac{1}{2}Tr_6TR\log\tilde{\Omega}_{ABCD}&=\int dX_{AB}e^{-TrX_{AB}\tilde{\Omega}_{ABCD}X_{CD}}\\&=\left[\int dX^{(1)}_a\ e^{-2TrX^{(1)}_a\tilde{\Omega}_{ab}X^{(1)}_b}\right]^2\\&=Tr_3TR\log\tilde{\Omega}_{ab}.\end{aligned} \tag{4.11}$$

The contributions coming from the two spheres are equal and hence the factor of 1 (instead of $\frac{1}{2}$) in front of the last logarithm. Tr_3 is the trace associated with the action of $SU(2)$ on the two dimensional sphere. The Laplacian $\tilde{\Omega}_{ab}$ is defined by

$$\begin{aligned}\tilde{\Omega}_{ab}&=2\mathcal{L}^2_{AB}\delta_{ab}+16\left(1-\frac{1}{\phi}\right)i\epsilon_{abc}\mathcal{L}^{(1)}_c+8m^2\tilde{\Omega}^{(1)}_{ab},\\\tilde{\Omega}^{(1)}_{ab}&=4P^{(1)}_{ab}-\frac{1}{c_2}\mathcal{L}^{(1)}_a\mathcal{L}^{(1)}_b+2\left(1-\frac{1}{\phi^2}\right)\delta_{ab}.\end{aligned} \tag{4.12}$$

$P^{(1)}_{ab}$ is the normal projector on the fuzzy sphere defined by $P^{(1)}_{ab}=x^{(1)}_a x^{(1)}_b$ where $x^{(1)}_a$ are the coordinate operators defined in (2.6) with $R_1=R_2=1$. The presence of this projector means in particular that we can not diagonalize in the polarization tensors basis. However, in order to have an idea of the phase structure of the model, we can expand around $m=0$. This approximation was more than sufficient in the two-dimensional case as discussed in great detail in [1]. Therefore it is convenient to separate the logarithm term as

$$\log\tilde{\Omega}_{ab}=\log\tilde{\Omega}^{(0)}_{ab}+\log\left(1+8m^2\left(\frac{1}{\tilde{\Omega}^{(0)}}\right)_{ac}\tilde{\Omega}^{(1)}_{cb}\right). \tag{4.13}$$

$\tilde{\Omega}^{(0)}_{ab}$ is clearly equal to $\tilde{\Omega}_{ab}$ when $m^2=0$. This operator can be trivially diagonalized in the vector polarization tensors basis $(\hat{Y}^{j_1M_1}_{l_1})_a$ on the first sphere tensor product the scalar polarization tensors basis $\hat{Y}_{l_2m_2}$ on the second sphere. Indeed by introducing the total angular momentum on the two-dimensional sphere $\mathcal{J}^{(1)}_a=\mathcal{L}^{(1)}_a+\theta^{(1)}_a$ where $\theta^{(1)}_a$ are the generators of $SU(2)$ in the spin 1 irreducible representation we can rewrite $\tilde{\Omega}^{(0)}_{ab}$ in the following form

$$\frac{1}{8}\tilde{\Omega}^{(0)}_{ab}=(\mathcal{L}^{(1)}_c)^2\delta_{ab}+(\mathcal{L}^{(2)}_c)^2\delta_{ab}-\left(1-\frac{1}{\phi}\right)[(\mathcal{J}^{(1)}_c)^2_{ab}-(\mathcal{L}^{(1)}_c)^2\delta_{ab}-2\delta_{ab}]. \tag{4.14}$$

Hence it is convenient to use the following expansion for the matrices $X^{(1)}_a$ in (4.11)

$$X^{(1)}_a=\sum_{j_1M_1\ell_1}\sum_{\ell_2m_2}q^{j_1M_1\ell_1}_{\ell_2m_2}\left(\hat{Y}^{j_1M_1}_{\ell_1}\right)_a\otimes\hat{Y}_{\ell_2m_2}. \tag{4.15}$$

Thus

$$Tr_3TR\log\tilde{\Omega}^{(0)}_{ab}=\sum_{\ell_1j_1\ell_2}(2j_1+1)(2\ell_2+1)\log\left[1-2\left(1-\frac{1}{\phi}\right)\frac{j_1(j_1+1)-\ell_1(\ell_1+1)-2}{\ell_1(\ell_1+1)+\ell_2(\ell_2+1)}\right] \tag{4.16}$$

In the limit $L\longrightarrow\infty$ it is easily verifiable (for example by making an expansion in $1-\frac{1}{\phi}$) that this term is subleading compared to L^4. The second contribution in the limit $m\longrightarrow 0$ is given by

$$\begin{aligned}Tr_3TR\log\left(1+8m^2\left(\frac{1}{\tilde{\Omega}^{(0)}}\right)_{ac}\tilde{\Omega}^{(1)}_{cb}\right)&\approx 32m^2Tr_3TR\left(\frac{1}{\tilde{\Omega}^{(0)}}\right)_{ac}x^{(1)}_cx^{(1)}_b\\&-\frac{8m^2}{c_2}Tr_3TR\left(\frac{1}{\tilde{\Omega}^{(0)}}\right)_{ac}\mathcal{L}^{(1)}_c\mathcal{L}^{(1)}_b+16m^2\left(1-\frac{1}{\phi^2}\right)Tr_3TR\left(\frac{1}{\tilde{\Omega}^{(0)}}\right)_{ab}.\end{aligned} \tag{4.17}$$

In the large L limit it is possible to show (see the appendix) that all terms in (4.17) are subleading compared to the L^4 behaviour seen in the second term in (4.7) and hence the full one-loop quantum contribution to the effective potential is given by the logarithmic potential in (4.7). Thus as long as we are in the region of the phase space near $m\approx 0$ the effective potential behaves in the large L limit as follows

$$\frac{V_{\rm eff}}{4L^4}=\frac{1}{4g^2L^2}\left(\frac{1}{4}\phi^4-\frac{1}{3}\phi^3+\frac{1}{4}m^2\left(\phi^2-1\right)^2\right)+\log\phi. \tag{4.18}$$

This result is to be compared with the quantum effective potential for $U(1)$ gauge fields on a single fuzzy sphere $\mathbf{S}^2_L$ computed in [1] which is given explicitly by

$$\frac{V_{\rm eff}}{L^2} = \frac{1}{2g^2}\left[\frac{1}{4}\phi^4 - \frac{1}{3}\phi^3 + \frac{1}{4}m^2(\phi^2-1)^2\right] + \log\phi. \tag{4.19}$$

$$V_{\rm eff}(\phi) = 2c_2N^2\alpha^4\left[\frac{1}{4}\phi^4 - \frac{1}{3}\phi^3\right] + 4c_2\log\phi + \text{subleading terms}. \tag{4.20}$$

5. The $\mathbf{S}^2_L\times\mathbf{S}^2_L$-to-matrix phase transition

The second term in the potential (4.18) is not convex. This implies that there is a competition between the classical potential and the logarithmic term which depends on the values of m and g. The equation of motion $\frac{\partial V_{\rm eff}}{\partial\phi}=0$ will admit in general two real solutions where the one with the least energy can be identified with the fuzzy $\mathbf{S}^2_L\times\mathbf{S}^2_L$ solution (3.4). This equation of motion reads

$$(1+m^2)\phi^4 - \phi^3 - m^2\phi^2 + 4g^2L^2 = 0. \tag{5.1}$$

The quantum solution is found to be very close to 1, viz

$$\phi = 1 - \frac{4g^2L^2}{1+2m^2} + O((g^2L^2)^2). \tag{5.2}$$

However this is only true up to an upper value of the gauge coupling constant g (for every fixed value of m) beyond which the equation of motion ceases to have any real solutions. At this value the fuzzy $\mathbf{S}^2_L\times\mathbf{S}^2_L$ collapses under the effect of quantum fluctuations and we cross to a pure matrix phase. In other words we can not define a gauge theory everywhere in the phase space. As we will see below when the mass m is sent to infinity it is more difficult to reach the matrix phase and hence the presence of the mass makes the fuzzy $\mathbf{S}^2_L\times\mathbf{S}^2_L$ solution (3.4) more stable.

The critical value can be computed by requiring that both the first and the second derivatives of the potential $V_{\rm eff}$ with respect to ϕ vanish. In other words, for every fixed value of m the critical point is defined at the point (g_*, m) of the phase space where we go from a bounded potential to an unbounded potential. Solving for the critical value we get the results

$$\phi_* = \frac{3}{8(1+m^2)}\left[1+\sqrt{1+\frac{32m^2(1+m^2)}{9}}\right], \tag{5.3}$$

and

$$2g_*^2L^2 = -\frac{1}{2}(1+m^2)\phi_*^4 + \frac{1}{2}\phi_*^3 + \frac{m^2}{2}\phi_*^2. \tag{5.4}$$

In the particular case of $m^2=0$ the critical value is

$$g_*^2L^2 = \frac{1}{2}\left(\frac{3}{8}\right)^3. \tag{5.5}$$

Extrapolating to large values of the mass ($m\longrightarrow\infty$) we obtain the scaling behaviour

$$g_*^2L^2 = \frac{m^2+\sqrt{2}-1}{16}. \tag{5.6}$$

In figure 1 we plot the phase diagram defined by this equation[1]. As we increase the value of the coupling constant g (for a fixed value of m^2) there exists a *critical point* g_* where the fuzzy $\mathbf{S}_L^2\times\mathbf{S}_L^2$ solution becomes unstable and thus the minimum (3.4) disappears. Similarly as the value of the mass squared m^2 increases (for a fixed value of the coupling constant g) there is a *critical point* m_*^2 where $\mathbf{S}_L^2\times\mathbf{S}_L^2$ collapses. Clearly the value of m_*^2 is found by inverting equation (5.6) , viz

$$m_*^2 = 16g^2L^2+1-\sqrt{2}. \tag{5.7}$$

Finally we remark that as the value of m^2 increases it is more difficult to reach the transition point, in fact when $m^2\longrightarrow\infty$ the critical value g_*^2 approaches infinity.

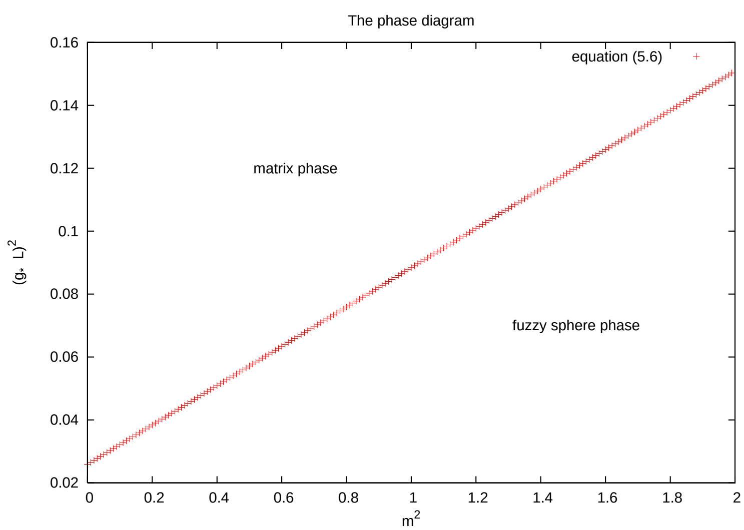

Figure 1: The $\mathbf{S}_L^2\times\mathbf{S}_L^2$-to-matrix critical line.

6. Conclusion

We have described the qualitative behaviour of a first order phase transition which occurs in a $U(1)$ gauge theory on $\mathbf{S}_L^2\times\mathbf{S}_L^2$. Using the one-loop effective potential (4.18) of this theory we found that there exists values of the gauge coupling constant g and the mass m for which the fuzzy $\mathbf{S}_L^2\times\mathbf{S}_L^2$ solution (3.4) is not stable. Thus for these values a $U(1)$ gauge theory on $\mathbf{S}_L^2\times\mathbf{S}_L^2$ is not well defined. This means in particular that the model

[1]Notice that if we allow m^2 to take negative values, the gauge coupling constant g_*^2 will be a more complicated function of m^2. However we are only interested in positive values of m^2 for which the behaviour of g_*^2 as a function of m^2 is the straight line (5.6) which can be deduced from the large m^2 behaviour of (5.3) and (5.4).

(3.2) can be used to approximate $U(1)$ gauge field theories on $\mathbf{S}^2 \times \mathbf{S}^2$ only deep inside the fuzzy sphere phase. However it is obvious from the critical line (5.6) that when the mass m of the two normal scalar fields on $\mathbf{S}^2_L \times \mathbf{S}^2_L$ goes to infinity it is more difficult to reach the transition line. Therefore we can say that our main goal of defining a nonperturbative regularization of a $U(1)$ gauge theory on $\mathbf{S}^2 \times \mathbf{S}^2$ is achieved. Generalization to $U(n)$ with and without fermions should be straightforward as long as we are only interested in the effective potential.

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A. Evaluation of I_1, I_2 and I_3.

In this appendix we show that the 3 terms in (4.17) are subleading compared to L^4. Let us define

$$I_1 = \frac{1}{L^4} Tr_3 TR \left(\frac{1}{\tilde{\Omega}^{(0)}}\right)_{ab}, \quad I_2 = \frac{1}{L^6} Tr_3 TR \left(\frac{1}{\tilde{\Omega}^{(0)}}\right)_{ab} \mathcal{L}_b^{(1)} \mathcal{L}_c^{(1)},$$
$$I_3 = \frac{1}{L^4} Tr_3 TR \left(\frac{4}{\tilde{\Omega}^{(0)}}\right)_{ab} x_b^{(1)} x_c^{(1)}. \tag{A.1}$$

We evaluate these traces by using the base of polarization tensors $\left(\hat{Y}^{j_1 M_1}_{\ell_1}\right)_a \otimes \hat{Y}_{\ell_2 m_2}$. Using the identity $\mathcal{L}_a \mathcal{L}_b = \mathcal{L}^2 \delta_{ab} - (\theta \cdot \mathcal{L})_{ab} - (\theta \cdot \mathcal{L})^2_{ab}$ the eigenvalues of the operator $\mathcal{L}_a \mathcal{L}_b$ are given by

$$\eta_{\ell_1 j_1} = \frac{1}{4} \left(j_1 \left(j_1 + 1\right) - \ell_1 \left(\ell_1 + 1\right)\right)^2 - \frac{1}{2} \left(j_1 \left(j_1 + 1\right) + \ell_1 \left(\ell_1 + 1\right)\right), \tag{A.2}$$

whereas the eigenvalues of $\tilde{\Omega}^{(0)}_{ab}$ given by (4.14) are

$$\lambda^{\ell_1 j_1}_{\ell_2} = 8 \left(\ell_1 \left(\ell_1 + 1\right) + \ell_2 \left(\ell_2 + 1\right)\right) + 8 \frac{1-\phi}{\phi} \left(j_1 \left(j_1 + 1\right) - \ell_1 \left(\ell_1 + 1\right) - 2\right). \tag{A.3}$$

Using these facts the two quantities I_1 and I_2 can be shown to be given by

$$I_1 = \frac{1}{L^4} \sum_{\ell_1 j_1 \ell_2} \frac{\left(2 l_2 + 1\right) \left(2 j_1 + 1\right)}{\lambda^{\ell_1 j_1}_{\ell_2}}, \qquad I_2 = -\frac{1}{L^6} \sum_{\ell_1 j_1 \ell_2} \frac{\left(2 \ell_2 + 1\right) \left(2 j_1 + 1\right) \eta_{\ell_1 j_1}}{\lambda^{\ell_1 j_1}_{\ell_2}}. \tag{A.4}$$

In order to evaluate I_3 we notice the fact that $x_a^{(i)}$ is proportional to $\left(\hat{\mathbf{Y}}_1^{00}\right)_a$ thus by using the algebra of vectorial polarization tensors we get the identity

$$\mathrm{Tr} \left\{ \left(\mathbf{Y}^{jM}_\ell \cdot \mathbf{Y}^{00}_1\right) \left(\mathbf{Y}^{00}_1 \cdot \mathbf{Y}^{+\, jM}_\ell\right) \right\} = (L+1)(2\ell+1) \left\{ \begin{array}{ccc} 1 & \ell & j \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\}^2. \tag{A.5}$$

The final result for I_3 is

$$I_3 = \frac{2}{L^4} \sum_{\ell_1 j_1 \ell_2} \frac{2l_2+1}{\lambda_{\ell_2}^{\ell_1 j_1}} \left[(L+1)\,(2\ell_1+1) \left\{ \begin{array}{ccc} 1 & \ell_1 & j \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\}^2 \right]. \tag{A.6}$$

The large L behaviour of I_1, I_2 and I_3 can be studied with the help of the different identities of [14]. The first sum in (A.4) diverges at most as L^2 in the continuum $L \to \infty$ limit and hence I_1 converges to zero as $1/L^2$. On the other hand the sum in I_2 behaves at most as L^4 thus the whole expression goes to zero as $1/L^2$. For I_3 we can check that the sum goes as L, i.e I_3 appraoches 0 as $1/L^3$.

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